INTRODUCTORY CALCULUS FROM THE VIEWPOINT OF NON-STANDARD ANALYSIS - DERIVATIVE OF SINE AND COSINE

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Abstract

This article exemplifies a novel approach to the teaching of introductory differential calculus using the modern notion of "infinitesimal" as opposed to the traditional approach using the notion of "limit". I illustrate the power of the new approach with a discussion of the derivatives of the sine and cosine functions.

1 OVERVIEW

Some teachers of mathematics may be surprised to learn that the concept of limits was a rather late (d'Alembert, 1754) addition to the underpinnings of the calculus. The inventors of calculus worked directly with concepts such as "infinitely little lines" (Newton, in Cohen and Westfall's *Newton*, Norton, New York 1995, p.381), presumably adopting the most intuitively congenial approach to the subject. This fact suggests that it is counter-intuitive to introduce students to the derivative as the outcome of a limiting process. In the words of one mathematics teacher:

> As someone who has spent a good portion of his adult life teaching calculus courses for a living, I can tell you how weary one gets of trying to explain the complex and fiddling theory of limits to wave after wave of uncomprehending freshmen.

(Rucker. *Infinity and the Mind*, Birkhäuser, Boston 1982, p. 87).

Newton and Leibniz, Leibniz especially, seem to have conceived of very small quantities, "infinitesimals", that lay outside of the ordinary number system. Leibniz spoke of the derivative of a function as being the ratio of two such infinitesimals. By the beginning of the twentieth century, however, the notion of "infinitesimal quantity" had disappeared from mathematics textbooks (although it persists in some physics and engineering texts as, for example, in the "principle of virtual work".). Then, in 1966, Abraham Robinson showed that the real number system can be extended to incorporate the notion of infinitesimal in approximately the sense of Leibniz (Robinson, *Non-standard Analysis*, North-Holland, Amsterdam 1966). While the foundations and rules developed by Robinson involve advanced mathematics (a readable account may be found in Stroyan and Luxembourg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York 1976) the results can be used to simplify greatly the solutions of many long-standing mathematical problems.

There are existing texts that incorporate non-standard analysis [1] These have failed, I believe, to clarify the subject for undergraduates because the authors apparently felt honor-bound to clutter their presentation of the intuitively accessible ideas by first presenting rigorous justifications of the mathematics.

I contend that the basic ideas are easily understood by students who are probably totally impervious to the justifications. I also contend that students will experience far less discomfort in dealing with infinitesimals from the outset than they presently experience while trying to absorb and utilize the notion of limits and related mathematical underpinnings.

I show in this article how to avoid the early introduction of concepts that, while important to the rigorous development of the subject of calculus, are totally incomprehensible to beginning students. The present approach is, instead, intended to build on students' intuitive concepts, with mathematical rigor generally left for advanced courses where rigor aids, instead of detracts from, the conceptual development.

The underlying idea (for example in treating difference quotients in finding the slope of a curve), is to treat an extremely small increment of the independent variable (dubbed the 'dibbl') as if its square were equal to zero. This idea gives one an occasion to exercise students on the powers of large and small numbers, leading to a single new concept for the student to confront. This approach, which allows one to defer the notion of limits, can be justified using non-standard analysis. The results can, of course, be duplicated using limits - often with much more work.

The approach illustrated in this article differs in another significant way from that found in existing calculus textbooks. Almost all such textbooks seem to assume that a student learns the subject in a logically linear fashion. First we learn about functions and continuity, then about limits, then the derivative, and so forth. The student isn't told, until somewhere

[1] *e.g.* H. Jerome Keisler, *Elementary Calculus, an Approach Using Infinitesimals*, Bodgen & Quigley, 1971, (2nd ed. entitled *Elementary Calculus: an Infinitesimal Approach* 1986), and Keith Stroyan's *Calculus Using Mathematica* (1993) (second edition entitled *Calculus, the Language of Change* 1997)

around chapter 4 of the typical text, that all of this mathematical apparatus is somehow related to solving problems concerning motion, velocity, acceleration, and related concepts.

The present approach is structured to take into account that the learning of mathematics is a highly non-linear process. Oftentimes the mathematical (or "logical") justification for a procedure cannot be understood by a student until long after the student has learned the procedure. All of us, I think, gain new insights into most topics in mathematics each time we revisit the topic. There is need, therefore, for texts that emphasize such revisitations as a part of the learning process.

Many introductory students are weak on algebra skills and have had minimal exposure to the concept of "proof". This article incorporates exercises, and a pedagogical approach, intended to assist a teacher in addressing such weaknesses. I try at all times to incorporate the pedagogical principles set out in Arnold Arons' book *A Guide to Introductory Physics Teaching* (Wiley, New York 1990). Although Arons' book ostensibly addresses the teaching of physics, much of the subject matter addresses means for remedying the mathematical deficiencies of typical engineering and science students. Also, I ordinarily prohibit, in keeping with this spirit, the use of calculators in working most (but not all) problems, accepting rough estimates of numerical values, where appropriate.

I adhere, for the most part, to the American Standard Abbreviations for Scientific and Engineering Terms as reproduced, for example, in *Handbook of Chemistry and Physics* (63d Ed., CRC Press, Boca Raton, FL., 1982)

Section 2 of the present article is a highly abbreviated introduction to differentiation, using the concept of the "dibbl", a quantity so small that its square is zero. The section omits, for the sake of brevity, many topics that I would include in the first six chapters of an introductory text such as the idea that "velocity" is the slope of the tangent to a distance-time curve, or that "rate" is the slope of the tangent to some other kind of curve. (See, *e.g.* excerpts from a proposed text at http://www.hep.anl.gov/jlu/index.html.) It also omits such vital topics as the product and chain rules.

Section 3 uses the dibbl method to obtain the derivatives of the sine and cosine functions. I believe that the derivation presented here is novel (for a textbook), and is much easier to understand and reproduce than the presentations in other texts. I deliberately omit, in that Section, any discussion of uniqueness. "Less is more", especially in introductory textbooks.

I simplify the language by speaking of the *slope of a curve* when I mean the *slope of the tangent to the curve.*

2 FINDING SLOPES

> Your Majesty says, "Kill a gentleman, and a gentleman is told off to be killed. Consequently, that gentleman is as good as dead - practically, he is dead - and if he is dead, why not say so? " *The Mikado, Act II*, by Gilbert and Sullivan

I motivate the derivative, as pointed out in the introduction, as measuring the rate of change of a quantity. We deal with quantities that can be represented by curves on a graph. Our problem is to find the rate of change of the quantity at a point on the graph. That rate of change is measured by the slope of the tangent at the point to the curve representing the function.

In what follows, I shall introduce a quantity, that I call a "dibbl", so small that its square "is as good as" zero - practically, it is zero - and if it is zero, why not say so? The dibbl will make it easier for us to calculate slopes.

2.1 NEW LANGUAGE

I show, in this Section, how to find the slopes of functions that can be described by integer powers of an independent variable. The problem is to find the equation of a straight line that is tangent to the function at some point. Once we know that equation we can just read off the slope of the line.

Let's try to find the slope of the tangent at some point, call it x_1, of the quadratic equation $y(x) = cx^2$ where c is some constant. Then the slope of a line joining that point to some other point on the curve, call it x_2, is just the rise over the run so that

$$\begin{aligned} \text{slope} &= \frac{y(x_2) - y(x_1)}{x_2 - x_1} = \frac{cx_2^2 - cx_1^2}{x_2 - x_1} \\ &= c(x_2 + x_1) \end{aligned} \tag{1}$$

We wanted to find the slope of the quadratic at a point whose $x-$ coordinate is equal to x_1. But the process of taking the slope involves two points because the slope is the rise of y between two points divided by the run along x between the same two points. So we have a result that depends, not only on the point with $x = x_1$, where we want the slope, but also on some other point with $x = x_2$. We have, in other words, the slope of a line that cuts the curve at two points.

But we only want the tangent line to touch the curve at the one point, namely where $x = x_1$. What to do? Choose x_2=x_1! Then Eq. 1 becomes:

$$\text{slope} = 2cx_1 \tag{2}$$

Eq.2 shows that it is easy to find the slope of a quadratic function $y(x) = x^2$ at a point $x = x_1$. The slope is just the ratio of the change in y to the change in x between two points x_2 and x_1 and then letting $x_2 = x_1$. Writing the last statement in symbols, we have

$$\text{slope} = \frac{y(x_2) - y(x_1)}{x_2 - x_1} \tag{3}$$

and then let $x_2 = x_1$. The same procedure should work for any smooth function $y(x)$, but the procedure would become pretty tedious if the function is very complicated, as the next exercises demonstrate.

EXERCISE

2.1 Find the slope of the function $y(x) = x^4$ at the point $x = x_1$ using the method of Eq. 3. Show all your work. Hint: (Check your result by noting that when $x_1 = 3$ the value of the slope is 108.)

2.2 You are running around a circular quarter-mile track. The coach tells you that your distance from the starting line is increasing proportionally to the cube of the time. At the end of 1 second you have traveled 3 m.

(a) Write an equation for $s(t)$, the distance traveled, as a function of t. Check your equation by noting that you must have traveled 24 m at the end of 2 seconds.

(b) Your speed at any time t is the rate of change of your distance at time t.
Find your speed at any time t. (Here we are writing just t, instead of t_1).

(c). A horse can gallop with a top speed of about 7 m/sec. After how many seconds are you running at a horse's top speed?

(d) Did you have to consider the shape of the track when you answered (a) through (c)?

It turns out that we can simplify the calculation of slope by taking the points x_2 and x_1 to be very close together. This makes sense, because at the end of the calculation we are going to let $x_2 = x_1$. So let's start by simplifying the notation, dropping the subscripts and restating the problem as follows: **Find the slope of the function $y(x)$ at some value of x.**

We will do this by finding the slope of a straight line that intersects $y(x)$ at two points that are so close together that they are **practically, but not quite, the same point**. We write the two points as $y(x)$ and $y(x + dx)$. Since x and $x + dx$ are nearly the same, we say that $x + dx$ is just a *dibbl* away from x.

Also, since we know that $y(x)$ is described by a smooth curve, we also know that the two points on the curve with y-values $y(x)$ and $y(x + dx)$) are just a dibbl away from each other.

I have already said that a dibbl is a quantity so small that we can take its square to be zero "for all practical purposes." So the rest of this article makes use of "the dibbl equation"

$$dx \times dx = dx^2 = 0 \tag{4}$$

The dibbl equation comes into play when we recalculate the slope of a quadratic at the point $y(x)$. We write, for $y(x) = cx^2$:

$$\begin{aligned} \text{slope} &= \frac{y(x+dx) - y(x)}{(x+dx) - x} \\ &= \frac{c(x^2 + 2cxdx + \overbrace{dx^2}^{=0}) - cx^2}{dx} \\ &= \frac{2cx \times \cancel{dx}}{\cancel{dx}} = 2cx \end{aligned} \tag{5}$$

Eq. 5 no longer requires the step of setting the point x_2 equal to the point x_1. The dibbl equation makes that step unnecessary.

Note that the slope in Eq. 5 is a function of x. It is, in fact, a function of x that was *derived* from the original function $y = cx^2$. Mathematicians, for this reason, call the slope of a function y(x) the *derivative* and write it as a ratio of two dibbls, dy and dx. To summarize, then, for any smooth function $y(x)$ there is a slope function, called the derivative,

$$\frac{dy}{dx} = \frac{y(x+dx) - y(x)}{(x+dx) - x} = \frac{y(x+dx) - y(x)}{dx} \tag{6}$$

IMPORTANT NOTE Many textbooks, for historical reason, write the derivative of a function $y(x)$ with a "prime" symbol. So the symbols $y', y(x)', y'(x)$, and $\frac{dy}{dx}$ all mean exactly the same thing. I shall sometimes use the prime symbol in this article, so that students can accustom themselves to both usages.

EXERCISE

2.3 Redo Exercise 2.1, this time using the dibbl method.

2.2 POWERS OF SMALL NUMBERS; THE DIBBL dx

We have introduced the dibbl as a quantity "that is so small that its square is as good as zero". "How small is that?", you might say. So let's get some experience with the squares of small numbers.

EXERCISE

2.4 You may use your calculator for this exercise:
(a) How much is the square of $\frac{1}{5}$? (Hint: If you didn't get $\frac{1}{25}$, ask your teacher to send you back

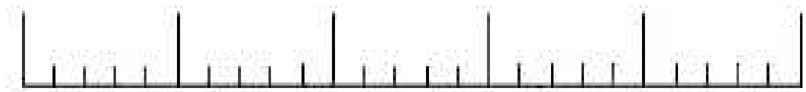


FIG 1

to third grade).
(b) Now take a piece of paper and a ruler marked in centimeters and draw a straight line that is 10 cm long. Divide the line into 5 equal parts by marking little perpendicular lines ("tick marks") on the line, like the longer vertical lines in Fig. 1.

Next, divide each division into 5 equal parts, again using tick marks. Into how many divisions have you divided the 10 cm line? Your line should resemble the line in Fig. 1.
(c) Does the distance between the little tick marks in Fig 1 represent the square of the distance between the large tick marks? (Hint: If you answered "No", go back and redo part (a) of this Exercise). Now complete the following table, using your calculator if you so desire:

$$\begin{array}{llllll}
x = & \frac{1}{5} = & .2 & x^2 = & \frac{1}{25} & = .04 \\
= & \frac{1}{50} = & & & & = \\
= & \frac{1}{500} = & & & & = \\
= & \frac{1}{5000} = & & & & = \\
= & \frac{1}{50000} = & & & & =
\end{array}$$

(d) Are you tired of writing out the zeros in part (c)? Let's do the same calculations in power-of-ten notation, which we will use in all future exercises that involve numbers:

$$\begin{array}{llllll}
x = & \frac{1}{5} = & 2 \times 10^{-1} & x^2 = & \frac{1}{25} & = 4 \times 10^{-2} \\
 & \frac{1}{5\times 10} = & & & & = \\
 & \frac{1}{5\times 10^2} = & & & & = \\
 & \frac{1}{5\times 10^3} = & & & & = \\
 & \frac{1}{5\times 10^5} = & & & & =
\end{array}$$

Exercise 2.3 shows that the square of a fraction means "taking a fraction of a fraction". If the fraction is a very small number then its square may be very small indeed. How small must the number be so that the square "is as good as" zero? In other words, how small is a dibbl? This is a profound question. It suffices to say for now that a dibbl is not an ordinary number, but is something that is smaller than any fraction!

We close this subsection by reminding ourselves why it is difficult to assign a value to $\frac{1}{0}$.

EXERCISE

2.5

(a) Ask your calculator to find a value for $\frac{1}{0}$, and write down the result. If you got a value, you are finished.

(b) If your calculator couldn't do the problem try dividing 1 by a lot of values in the denominator that are close to zero. Then you can try to guess the result. One way to do this is to take a sheet of graph paper, and plot values of $\frac{1}{x}$ for a lot of different fractional values of x. (Don't forget to include negative values). The answer should be roughly half way between the value for the smallest positive value of x and the smallest negative value. What is your result?
(c) If you are in a study group, compare your result with others in your group. Can you all agree on the same value?
(d) What would you say to someone who claims that division by zero is "undefined"? Explain your answer, referring to your results in part (b).

2.3 dx^2 IS ZERO SIMPLIFIES THE BINOMIAL THEOREM

EXERCISE

2.6 (a). Write down the derivative functions (the slope function, remember?) $\frac{dy}{dx}$ for $y(x) = x, x^2, x^3, x^4$. (You worked these out in Subsection 2.1).
(b) Guess what the derivative function is for $y(x) = x^n$ where n stands for any positive integer (do not use a specific numerical value for n), and write down your answer.

We now help you work out the answer to Exercise 2.5(b), using the definition of the derivative in Eq. 6. The first step is to expand the product $(x + dx)^n$, remembering that all powers of dx higher than the first power are zero. We write:

$$y(x + dx) = (x + dx)^n = \overbrace{(x + dx)(x + dx)\cdots(x + dx)}^{\text{n times}} \tag{7}$$

The highest power of x in Eq. 7 comes from multiplying all of the $x's$ together. There is only one way to do this, so there is a leading term x^n.

The next term will come from multiplying together $(n-1)$ $x's$ and one dx. How many such terms are there? That's an easy question, because each term omits one dx and there are exactly n different $dx's$ in the product in Eq. 7. The second term is therefore $nx^{n-1}dx$.

What about the next term? It will have the product of $(n-2)$ $x's$, two $dx's$ and a numerical coefficient, let's call it a. There will be more terms, involving products of three and even more $dx's$ (if n is bigger than 2). The numerical coefficient that goes with each term is known from the *binomial theorem* that people learn about in pre-calculus courses.

But we don't care about the value of a and all those numerical coefficients because they all involve products of two or more $dx's$, and since dx is a dibbl the product of two or more $dx's$ is zero. The result, then, is $(x + dx)^n = x^n + nx^{n-1}dx$, which is the statement of the

binomial theorem for x plus a dibbl taken to any power n.

Eq 6 gives, for $y(x) = x^n$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^n)}{dx} \\ &= \frac{(x^n + nx^{n-1}dx) - x^n}{dx} \\ &= \frac{nx^{n-1}dx}{dx} = nx^{n-1} \end{aligned} \tag{8}$$

Eq 8 is often called "the power law".

EXERCISE

2.7 Calculate the derivatives $y' \equiv \frac{dy}{dx}$ ($\equiv$ means that the two expressions mean the same thing), using the method of Eq 8 when
(a) $y(x) = x^9$;
(b) $y(x) = 5x^{17}$;
(c) $y(x) = 6x^5 + 4x^4$
Show all of your work.

2.4 "DERIVATIVE" IS A FANCY NAME FOR VELOCITY OR RATE OF CHANGE

EXAMPLE 2.1

Find an expression for the straight line that is tangent to the curve described by $y(x) = \frac{1}{7}x^5$ at the the point $x = 2$. Show your work and graph the result.
Solution:
(1) Let $y_t(x) = a + bx$ be the equation for the straight line. Then our job is to determine the values of the constants a and b.
(2) Since $y_t(x)$ is a tangent, it must have the same slope as $y(x)$ at $x = 2$. But, as you learned in algebra, the slope of $y_t(x)$ is b. The slope of $y(x)$ at any point x is $\frac{dy(x)}{dx}$
(3) At $x = 2$, $\frac{dy(x)}{dx} = \frac{5}{7}x^4 = \frac{80}{7}$. So $b = \frac{80}{7}$.
(4) Also, $y_t(x)$ and $y(x)$ must be equal at the point of tangency, since they share the same point. So $a + \frac{80}{7}x = \frac{1}{7}x^5$ at $x = 2$, or $a + \frac{160}{7} = \frac{32}{7}$. Thus, $a = -\frac{128}{7}$.
(5) The solution is $y_t(x) = \frac{-128+80x}{7}$. Fig. 2 is a graph of the solution showing the tangent line and the curve.

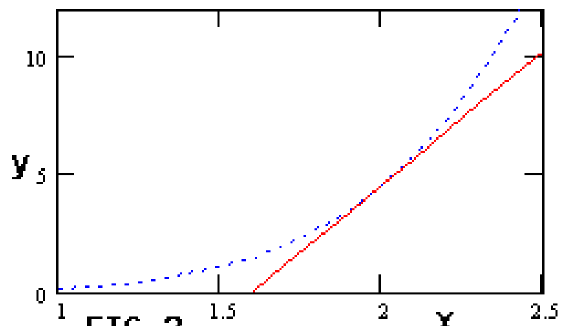


FIG 2

Recall why we are interested in the slopes of tangents to curves. We began our discussion of calculus by demonstrating that “the slope of a line on a distance-time graph . . . corresponds to velocity”. So if we are given a function that relates distance to time, the derivative of that function, being the the slope of the function, corresponds to the velocity as a function of time. The next exercise makes use of that relationship.

EXERCISE

2.8 (a) A rock is hurled upward from a platform that is H meters (Hm) above the ground with an initial velocity $v_0\ m/sec$. We learn in physics that the height of the rock above the ground at time t after it left the platform is

$$h(t) = H + v_0 t - ct^2 \tag{9}$$

where c is a constant. Find the velocity function $v(t)$ of the rock. (Hint: use the method of the previous subsection. Make sure that you treat "h" and "H" as two different symbols.)
(b) Find an expression for t_m, the time when the velocity function equals zero.
(c) Find an expression for the slope of $h(t)$ at time t_m.
(d) Find an expression for the straight line that is tangent at time t_m to the curve described by $h(t)$ on a distance-time graph.
(e) Find an expression for the maximum value of $h(t)$ as a function of t. (Hint: to check your work, the answer to this part, when $H = 8\ m$; $v_0 = 12\ m/sec$; and $c = 5\ m/sec^2$, is $15.2\ m$.
(f) Make a graph of $h(t)$ on the vertical axis against t on the horizontal axis, using the values in part (e). The horizontal axis ofyour graph should be at least 15 cm long, and the horizontal scale should range from 0 to 4 secs. Calculate $h(t)$ for at least 8 values of t.

2.9 Now let's work out the equation for the tangent line to any power function. So let $y(x) = cx^n$ where c can be any constant and n is any positive integer. Write the tangent line, as before, $y_t(x) = a + bx$.

(a) Find the value of b at an arbitrary point x_0. (Hint: b will be a function of $c, n, and x_0$).
(b) Now find the value of a and write your expression for $y_t(x)$.
(c) Check your result by letting c and n take the values used in the example.

2.10 And, for an algebra brush-up, solve for t by completing the square $at^2 + bt + c = 0$. Show all of your work.

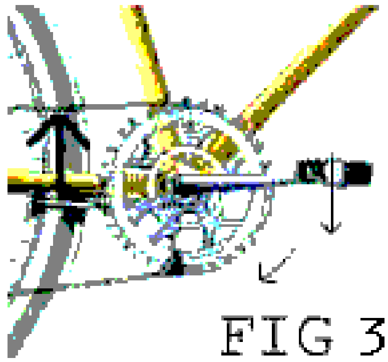


FIG 3

I OMIT, FOR THE SAKE OF BREVITY, SECTIONS ON THE PRODUCT AND CHAIN RULES

3 TRIGONOMETRIC FUNCTIONS

> ...they be crammin my mind with [stuff] they want us to remember, ... all kinds of crap like coaxiel coordinates, cosine computations, spheriod trigonometry, ...
>
> -Winston Groom (*Forrest Gump*)

In this section I present a novel (for an introductory textbook) discussion of the derivatives of the sine and cosine functions. Many students at this level have not learned to view the trigonometric functions as functions. I therefore begin by introducing that viewpoint. "Spheriod trigonometry", however, can safely be left for a later course.

3.1 Circular Motion

The sine and cosine can be used in the description of circular motion. Look, for example, at Fig 3 which is a close-up view of the working part of a bicycle. The bicycle moves forward as the rear wheel is forced to rotate by the chain, which is driven by the rotation of the sprocket. But the sprocket is made to rotate by the up-and-down motion of the bicyclist's feet.

Fig 4 is a graph that plots the vertical position of the bicyclist's left foot against time, assuming that the bicyclist rotates the sprocket with constant speed. The vertical scale shows the height of the bicyclist's left foot above or below the position shown in Fig. 3, which is taken to be at zero time.

The curve shown in Fig. 4 represents the sine function that you learned about in connection with right triangles. Its connection with circular motion should become clear after the next exercise.

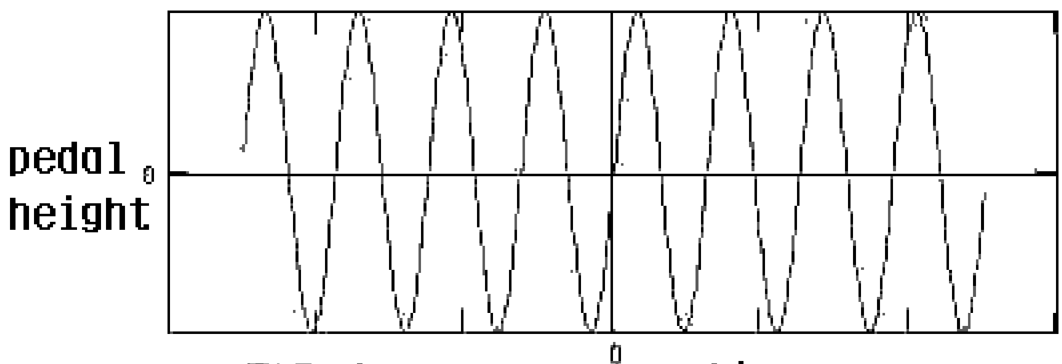


FIG 4

EXERCISE

3.1

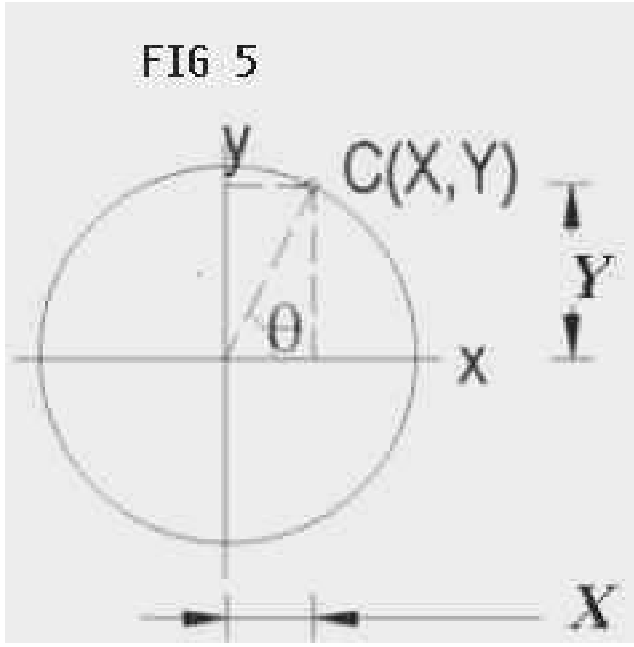
(a) The circle shown in Fig. 5 has a radius that is exactly equal to one unit of length. Express the distances X and Y, shown in the figure, in terms of the sine and cosine of the angle θ. (Hint: if you have trouble with this part, review the sine and cosine definitions in one of your old math books - or a dictionary.)
(b) Make a graph of the length Y as a function of the angle θ, using θ intervals of 10° between 0 and 360°. Does your graph resemble a portion of Fig. 4? Be sure to plot the Y values on the vertical axis.
(c) Suppose the point C in the figure moves around the circumference of the circle with constant speed so that $\theta = pt$, t being the time. Can you now make a graph of the length Y as a function of the time? (Hint: Can you save yourself a lot of work be just relabeling the x-axis of your previous figure? Does it help to use time units of $\frac{degrees}{p}$, even though you don't have a numerical value for p?
(d) Now sketch, without calculating, the graph of Y for one more complete revolution of the point C around the wheel. Compare your graph with Fig. 3. Explain in one or two sentences what your graph would look like if it showed Y-values for many revolutions of the wheel.

The last exercise demonstrates that the vertical position of the point C in Fig. 5 can be written as a function of time (assuming that the sprocket rotates with constant speed) as

$$Y(t) = R\sin(pt) \tag{10}$$

where p is the rotational speed of the sprocket and R is the radius of the circle described by the pedal. Similarly the horizontal position of C would be written as

$$Y(t) = R\cos(pt) \tag{11}$$



The vertical positions of the pedals in Fig. 3 would be equal to $Y(t)$, as Fig. 4 indicates.

Let's get more familiar with the sine function.

EXERCISE

3.2

Suppose that a function $y(t) = R\sin(pt)$, where R and p are constants. You may use Fig. 4 as a graph of $y(t)$.

(a) Suppose that pt is equal to some fixed number θ that lies between $0°$ and $360°$. How often will $y(t)$ repeat the same value of y as t increases forever?

(b) How many degrees separate the points on a graph of $y(t)$ as a function of pt where y(t) not only has the same value, but has positive slope? (Hint: How many degrees of rotation bring the point C back to the position shown in Fig. 5?)

(c) What are the values of $y(t)$ at the points where $y(t)$ has zero slope?

(d) Suppose that $x(t) = R\cos(pt)$. What are the values of $x(t)$ at the points where $y(t)$ has zero slope?

(e) What are the values of $y(t)$ at the points where $x(t)$ has zero slope?

(f) From the fact that $\cos(\theta) = \sin(\theta + 90°)$, what can you say about the *shape* of the graph of $x(t)$? (Hint: could you match the graphs of $x(t)$ and $y(t)$ by sliding one over the other?)

(g) Sketch the derivative of the curve shown in Fig. 3. Where the curve is increasing, the derivative is positive, the faster the increase then the more positive the derivative; correspondingly for places where the curve is decreasing. Compare your curve with the cosine curve of part (f). Redo your sketch after you have finished this subsection.

The functions $x(t)$ and $y(t)$ in the last exercise keep repeating themselves as t increases. The repetitious behavior is not surprising, because it is connected with the motion of a point that is moving around the same circle, over and over again, with constant speed.

We close this subsection with two little exercises on trigonometry.

EXERCISES

3.3

(a) A flagpole that has a circumference of w inches is planted on level ground. You are standing H feet away from the center of the base of the flagpole. You find that the line-of-sight from yourself to the top of the flagpole makes an angle of θ degrees with the ground. Your eyeballs are h feet above the ground. Find an expression for the height of the flagpole above the ground.
(b) Check your answer by noting that when $\theta = 30°$, $H = 200$ and $h = 5$, the height is about 120 feet.
(c) How much does your answer change if the circumference is doubled?

3.4

Recall that the addition formula for the sine function is

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{12}$$

Suppose that the symbol θ stands for some fixed angle. List all of the angles $\theta + \alpha$ for which the graph of the sin function looks exactly the same as it does at θ. (Hint: a possible wrong answer would be $\theta + n \times 180°$, where n is any positive or negative integer.)

3.2 DERIVATIVE OF THE PYTHAGOREAN THEOREM

3.2.1 Derivative of the Sine and Cosine

You showed, in part (a) of Exercise 3.1 that the distances X and Y are respectively equal to $\sin\theta$ and $\cos\theta$. The radius of the circle in Fig. 5, you were told, is 1 unit so, by the Pythagorean theorem, $X^2 + Y^2 = 1$. Expressing X and Y in terms of $\sin\theta$ and $\cos\theta$ gives the familiar relation from trigonometry

$$\sin^2\theta + \cos^2\theta = 1 \tag{13}$$

EXERCISES

3.4

Suppose that the radius of the circle in Fig. 5 is different from 1 - call it r, where r is some fixed length. What would then be the sum of the squares of the $\sin$ and $\cos$? (Hint: divide the

Pythagorean theorem by r^2.)

3.5
Take the derivative of both sides of the equation

$$\sin^2\theta + \cos^2\theta = 1 \tag{14}$$

Since you don't know the derivative of either $\sin$ or $\cos$ you may respectively write these as $\sin'\theta$ and $\cos'\theta$. (Hint: use the chain rule and the product rule).

The derivative of the Pythagorean Theorem (be sure to do the last exercise), expressed in terms of the sin and cos, can be rearranged so that it reads

$$\frac{\sin'\theta}{\cos\theta} = -\frac{\cos'\theta}{\sin\theta} \tag{15}$$

This equation contains two unknown functions, $\sin'\theta$ and $\cos'\theta$. Equations of this sort, containing unknown derivative functions are called *differential equations*. They are often solved by guessing the solutions, as we now illustrate.

Eq 15 is an equality between two ratios, each involving an unknown derivative function. One possible guess is that each ratio is equal to a constant, call it A. If this were true then each derivative would itself be proportional to a trigonometric function. Would this be a sensible result?

We know that both the sin and cos are functions that keep repeating themselves at regular intervals (intervals of 360°; such functions are called *periodic* functions with period 360°). We also know that the *slope* of the sine function is zero at each point where the *value* of the cosine function is zero, and *vice versa*.

It is obvious that the slope function for the sin is a periodic function that has a zero value at each point where the slope of the sin is zero. Surely the cos is such a function. So our guess is that

$$\sin'\theta \equiv \frac{d\sin\theta}{d\theta} = A\cos\theta \tag{16}$$

where A is some constant.

EXERCISE

3.6
(a) Obtain an equation for $\cos'\theta$. (Hint: you should be able to make use of the argument in the preceding paragraphs.)
(b) Use the discussion of Exercise 3.2(f) to argue, in two or three sentences, that the guesses for $\sin'$ and $\cos'$ are reasonable. Use sketches to support your argument.

The guess works! Solutions to the differential equation are given by Eq 16 and

$$\cos' \theta \equiv \frac{d}{d\theta} \cos \theta = -A \sin \theta \tag{17}$$

But what about that constant A?

The first clue as to the meaning of the constant is that we have never, in the preceding equation, specified the units that we are using to measure the angle θ. We will explore this point further in the next subsubsection, taking advantage of the fact that your calculator can calculate trigonometric functions in different kinds of units.

3.2.2 The Constant A

EXERCISE

3.7

This is a calculator exercise. Estimate the slope of the sine function at $0°$ using two different sets of units, and using Eq. 3 of Section 2, as follows:
(a) Take $x_2 = 5°$, $x_1 = 0$. Make certain that your calculator is set to calculate in degrees. What is the value of A for this case (please, no more than 3 significant digits)?
(b) Engineers sometimes measure angles in units called grads. A grad is $\frac{1}{400}$ of a circle, so that $5° = 5.55\ grads$. Set your calculator on "grads" and repeat the calculation of part (a) to obtain an estimate of the value of A for the grad units.
(c) In part (b), did the value of the sine function change when you changed units? Can the value of the sine of an angle depend on the units that you use to measure the angle? (Hint: recall the definition of the sine as the ratio of lengths of a side and the hypotenuse of a right triangle).

If you did your calculation correctly, you should have found that slopes of the sin and cos functions are about 10% bigger (in magnitude) in degree units than they are in grad units. So why don't we simplify matters by choosing an angular measure for which $A = 1$?

What kind of units for measuring angles give $A = 1$? We can investigate this question by looking at the sine of a very small angle, namely, an angle that is a dibbl.

EXERCISE

3.8

(a) Write out the definition of the derivative in the expression from Eq 16, with $A = 1$,

$$\frac{d \sin \theta}{d\theta} \equiv \frac{\sin(\theta + d\theta) - \sin \theta}{d\theta} = \cos \theta,$$

but set θ equal to 0 on both sides of the equation.
(b) Solve the equation that you obtained in part (a) for $\sin(d\theta)$ to obtain an expression for the sine of a very small angle. Be sure to use the correct values of $\sin$ and $\cos$ at $\theta = 0$. Note that

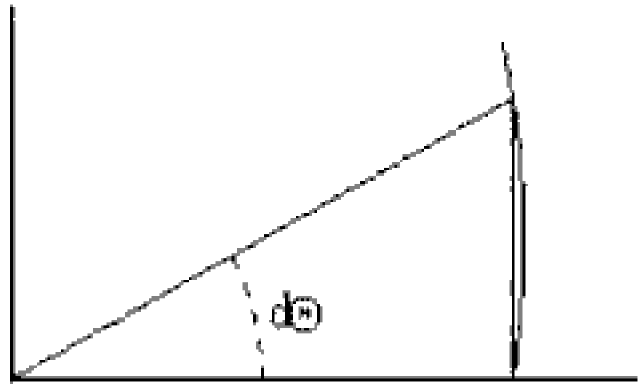


FIG 6

the $\sin$ of a dibbl is proportional to a dibble, and, therefore, is also a dibbl.
(c) Now find $\cos d\theta$ (**Hint**: recall that the square of a dibbl is zero).

The sine of a very small angle, expressed in units where $A = 1$ is just equal to the angle itself, as you have just demonstrated. That is, $\sin(d\theta) = d\theta$ in this special system of angular units. What are these special units?

Since, as you have also shown, the cosine of a dibbl angle is just 1, the side adjacent the angle is equal to the hypotenuse. The side opposite, therefore, begins and ends on the arc of a circle of radius equal to the hypotenuse.

In other words, the length of the side opposite the dibbl angle is just equal to the length of the arc swept out by the angle, as is shown in Fig. 6 (which exaggerates the curvature of the arc). But the sin is the length of the side opposite divided by the hypotenuse, which is the radius of the arc.

Conclusion: the angular measure corresponding to $A = 1$ is the arc length divided by the radius. You should recognize this as the definition of radian measure.

We end with an exercise that invites you to practice some of the skills you have developed thus far.

EXERCISE

3.9

Find the slope function for the function

$$F(t) = (\frac{t^{\frac{5}{3}}}{5 + 6t^{\frac{5}{3}}})^{\frac{2}{7}} \tag{18}$$

Check your answer by finding the slope at $t = 2$ where the slope is (to 3 significant digits) .0134.

3.3 ...AND, IN CLOSING

Trigonometry was important to early astronomers who, we blush to say, sometimes received income for giving astrological predictions. The sine of an angle had probably received its present definition by about the fifth century A.D. The Hindu mathematician Ãryabhata (born in 476) is credited with this development.

The study of the sin and cos as mathematical functions grew when the development of the calculus excited the interests of 17^{th} and 18^{th} century mathematicians. The Swiss mathematician Leonhard Euler, of whom you will hear much more in your studies of mathematics, may have been the first to treat the sine and cosine as mathematical functions and knew their derivative functions.

Euler, in fact, wrote the derivative of the sin in the form of a differential equation

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \tag{19}$$

You don't recognize it? Invert each side, substitute an angle θ for the variable y and $\sin\theta$ for the variable x. A little algebra will give you the result of this subsection, for the case where θ is measured in radians.

Now redo Exercise 3.2(g).

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